

On Capitulation of S -Ideals in \mathbf{Z}_p -Extensions

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Let k be a finite extension of \mathbf{Q} and p a prime number. Let K be a \mathbf{Z}_p -extension of k and S the set of all prime ideals in k which are ramified in K . We denote by A'_∞ the p -Sylow subgroup of the S -divisor class group of K . We give a criterion for $A'_\infty = 0$ which can be applied for general \mathbf{Z}_p -extensions. Further, we especially investigate the criterion for a totally real number field k in which p splits completely. © 2001 Academic Press

1. INTRODUCTION

Let k be a finite extension of \mathbf{Q} and p a prime number. Let K be a \mathbf{Z}_p -extension of k and $k_n \subset K$ the unique cyclic extension of k of degree p^n . Further, let S be the set of all prime ideals in k which are ramified in K . By Theorem 1 in [11], all prime ideals in S lie above p . We assume that all prime ideals in S are fully ramified in K . We denote by A_n the p -Sylow subgroup of the ideal class group of k_n . We put $A_\infty = \varinjlim A_n$, where the map: $A_n \rightarrow A_m$ is induced by the natural inclusion map $i_{n,m}: k_n \hookrightarrow k_m$ for $m \geq n$. We will denote the induced maps by $i_{n,m}$. Similarly, we denote by A'_n the p -Sylow subgroup of the S -ideal class group of k_n and put $A'_\infty = \varinjlim A'_n$.

The main purpose of this paper is to investigate the capitulation of S -ideals $H'_n = \text{Ker}(i_{n,\infty}: A'_n \rightarrow A'_\infty)$. For totally real fields k and $K = k_\infty$ the cyclotomic \mathbf{Z}_p -extension, some criteria for $A'_\infty = 0$, i.e., $A'_n = H'_n$ for all n , were given in [2, 4–6, 20]. We first generalize them to apply for general

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number fields k and general \mathbf{Z}_p -extensions. Put $S_n = \{\mathfrak{p}_n \mid \mathfrak{p}_n^{p^n} = i_{0,n}(\mathfrak{p}), \mathfrak{p} \in S\}$. Denote by k_{n,\mathfrak{p}_n} the completion of k_n at \mathfrak{p}_n and by $U_{\mathfrak{p}_n}$ the group of principal units in k_{n,\mathfrak{p}_n} . We define the following groups (cf. [19]):

$$U_n = \left\{ (u_{\mathfrak{p}_n}) \in \prod_{\mathfrak{p}_n \in S_n} U_{\mathfrak{p}_n} \mid \prod_{\mathfrak{p}_n \in S_n} \left(\frac{u_{\mathfrak{p}_n}, k_m/k_n}{\mathfrak{p}_n} \right) = 1 \text{ for all } m \geq n \right\},$$

$$V_{\mathfrak{p}_n} = \bigcap_{m \geq n} N_{k_m, \mathfrak{p}_m/k_n, \mathfrak{p}_n} U_{\mathfrak{p}_m}, \quad V_n = \prod_{\mathfrak{p}_n \in S_n} V_{\mathfrak{p}_n},$$

$$W_{\mathfrak{p}_n} = \bigcap_{m \geq n} N_{k_m, \mathfrak{p}_m/k_n, \mathfrak{p}_n} k_{m,\mathfrak{p}_m}^\times, \quad W_n = \prod_{\mathfrak{p}_n \in S_n} W_{\mathfrak{p}_n},$$

where $(\frac{u, k'/k}{\mathfrak{p}})$ is the norm residue symbol. Let d_n be the diagonal map $k_n^\times \rightarrow \prod_{\mathfrak{p}_n \in S_n} k_{n,\mathfrak{p}_n}^\times$. Let E_n be the group of units in k_n and E'_n the group of S -units in k_n . Define

$$\bar{E}_n = \overline{U_n \cap d_n(E_n)} \quad \text{and} \quad \bar{E}'_n = \overline{U_n \cap (d_n(E'_n) W_n)},$$

where \bar{A} is the topological closure of A .

THEOREM 1. *The following statements are equivalent.*

- (1) $A'_\infty = 0$.
- (2) $A'_0 \cong H^1(k_n/k, E'_n)$ and $U_n = V_n \bar{E}'_n$ for some n .

For every totally real number field k and the cyclotomic \mathbf{Z}_p -extension k_∞ , it is conjectured that $\#A_n$ is bounded as $n \rightarrow \infty$, which is equivalent to $A_\infty = 0$ (see [5, 11]). If Leopoldt's conjecture is valid for k and p , i.e., $(\mathbf{Z}\text{-rank of } E_0) = (\mathbf{Z}_p\text{-rank of } \bar{E}_0)$, the conjecture is equivalent to $A'_\infty = 0$. Several authors gave sufficient conditions for the conjecture and verified them for $p=3$ and quadratic fields with small discriminants (see [2, 4, 7, 8, 13]). However, the conjecture is not proved, in general. Following [5], we study two typical cases. (A) Only one prime ideal in k ramifies in K . (B) k is a totally real number field in which p splits completely, and Leopoldt's conjecture is valid for k and p . By studying inflation maps $H^2(k_n/k, E'_n) \rightarrow H^2(k_m/k, E'_m)$, we can show a difference between (A) and (B). The following corollary and theorem are reformulations of Theorems 1 and 2 in [5].

COROLLARY 1. *Assume (A). The following statements are equivalent.*

- (1) $A'_\infty = 0$.
- (2) $A'_0 \cong H^1(k_n/k, E'_n)$ for some n .

This corollary is immediately obtained from Theorem 1. In contrast to the former result, Theorem 1 in [5], we do not have to assume that k is totally real. For an extension M/L and a subgroup A of M^\times , we define

$$R(M/L, A) = \text{Ker}(H^2(M/L, A) \rightarrow H^2(M/L, M^\times)).$$

Let j_n be the natural map $R(k_n/k, E_n) \rightarrow R(k_n/k, E'_n)$. For a \mathbf{Z} -module A , put $\text{rk}_p A = \dim_{\mathbf{F}_p}(A/pA)$. We denote by $m \gg n$ that $m - n$ is sufficiently large.

THEOREM 2. *Assume (B). The following statements are equivalent.*

- (1) $A'_\infty = 0$.
- (2) $A'_0 \cong H^1(k_n/k, E'_n)$ for some n and $\text{rk}_p R(k_m/k, E'_m) = \text{rk}_p(R(k_m/k, E'_m)/j_m(R(k_m/k, E_m)))$ for all $m \gg 0$.

If (1) holds, the last statement can be verified by finite steps. We will give an example which explains how to apply (2) for the verification of (1).

For a general number field k , let \tilde{k} be the composite of all \mathbf{Z}_p -extensions of k and $L_{\tilde{k}}$ the maximal unramified abelian p -extension of \tilde{k} . Greenberg conjectured that $\text{Gal}(L_{\tilde{k}}/\tilde{k})$ is pseudo-null as a $\mathbf{Z}_p[[\text{Gal}(\tilde{k}/k)]]$ -module. When k is totally real and Leopoldt's conjecture is valid for k and p , this is equivalent to the above conjecture. For this generalized conjecture, capitulation of S -ideal classes in \mathbf{Z}_p -extensions is also important (cf. [15, 17]). We hope our criterion will play some role in the study of multiple \mathbf{Z}_p -extensions.

2. GENERAL CASE

We use the same notation as in the Introduction. Put $\Gamma = \text{Gal}(K/k)$ and $\Gamma_n = \text{Gal}(K/k_n)$. Fix a topological generator γ of Γ and put $\gamma_n = \gamma^{p^n}$. We denote by $N_{m,n}$ the norm map $k_m \rightarrow k_n$ for $m \geq n$. We will denote induced maps by $N_{m,n}$. Put $s = \#S = \#S_n$ and $H_n = \text{Ker}(i_{n,\infty}: A_n \rightarrow A_\infty)$. For a G -module A , we denote by A^G the fixed subgroup by all elements in G .

The following proposition was proved by Greenberg (see Proposition 2 and the proof of Theorem 1 in [5]).

PROPOSITION 1. *The following statements are equivalent.*

- (1) $\#A_n$ is bounded as $n \rightarrow \infty$.
- (2) $A_n = H_n$ for all $n \geq 0$.
- (3) $A_n^\Gamma \subseteq H_n$ for all $n \geq 0$.
- (4) $A_\infty = 0$.

In a similar way, we can prove the following proposition.

PROPOSITION 2. *The following statements are equivalent.*

- (1) $\#A'_n$ is bounded as $n \rightarrow \infty$.
- (2) $A'_n = H'_n$ for all $n \geq 0$.
- (3) $A_n'^F \subseteq H'_n$ for all $n \geq 0$.
- (4) $A'_\infty = 0$.

Let D_n be the subgroup of A_n consisting of classes which contain ideals whose prime divisors lie above S . Then we have $A'_n = A_n/D_n$. Let I_n be the ideal group of k_n , P_n the principal ideal group of k_n , and $Q_n = \{a \in I_n \mid \text{all prime divisors of } a \text{ lie above } S\}$. Put $D_{m,n} = ((Q_m i_{n,m}(P_n))/i_{n,m}(P_n))[p]$ and identify A_n with $(i_{n,m}(I_n)/i_{n,m}(P_n))[p]$, where $A[p]$ is the p -Sylow subgroup of A .

LEMMA 1. *There are exact sequences,*

$$0 \rightarrow H^1(k_m/k_n, E_m) \rightarrow D_{m,n}A_n \rightarrow A_m'^F \rightarrow R(k_m/k_n, E_m) \rightarrow 0,$$

$$0 \rightarrow H^1(k_m/k_n, E'_m) \rightarrow A'_n \rightarrow A_m'^F \rightarrow R(k_m/k_n, E'_m) \rightarrow 0.$$

Furthermore,

$$H^1(k_m/k_n, E_m) \cong E_m[N_{m,n}]/E_m^{\gamma_n-1},$$

$$R(k_m/k_n, E_m) \cong (E_0 \cap N_{m,n}k_m^\times)/N_{m,n}E_m,$$

$$H^1(k_m/k_n, E'_m) \cong E'_m[N_{m,n}]/E_m'^{\gamma_n-1},$$

$$R(k_m/k_n, E'_m) \cong (E'_0 \cap N_{m,n}k_m^\times)/N_{m,n}E'_m,$$

where $A[N_{m,n}] = \text{Ker}(A \rightarrow A \ (a \mapsto N_{m,n}a))$.

Proof. We obtain the above exact sequences from the p -part of seven term exact sequences independently found by Auslander and Brumer and Chase *et al.* (see [1, 12]). For a $\text{Gal}(k_m/k_n)$ -module A , since $\text{Gal}(k_m/k_n)$ is cyclic, we have

$$H^1(k_m/k_n, A) \cong A[v_{n,m}]/A^{\gamma_n-1} \quad \text{and} \quad H^1(k_m/k_n, A) \cong A[\gamma_n-1]/A^{v_{n,m}},$$

where $v_{n,m} = \sum_{i=0}^{p^{m-n}-1} \gamma_n^i$, $A[v_{n,m}] = \text{Ker}(A \rightarrow A \ (a \mapsto a^{v_{n,m}}))$, and $A[\gamma_n-1] = \text{Ker}(A \rightarrow A \ (a \mapsto a^{\gamma_n-1}))$. Since $N_{m,n}a = a^{v_{n,m}}$ and $A^F = A[\gamma_n-1]$, the lemma follows. ■

LEMMA 2. *For $\varepsilon \in E'_n$, $\varepsilon \in N_{m,n}k_m^\times$ if and only if $d_n(\varepsilon) \in W_n U_n^{p^{m-n}}$.*

Proof. For a prime ideal \mathfrak{q}_n of k_n which is not contained in S_n , \mathfrak{q}_n does not ramify in k_m . Hence, by local class field theory, ε is a local norm from the completion of k_m at any prime ideals lying above \mathfrak{q}_n . For $\mathfrak{p}_n \in S_n$, ε is a local norm from k_{m, \mathfrak{p}_n} if and only if $\varepsilon \in W_{\mathfrak{p}_n} U_{\mathfrak{p}_n}^{p^{m-n}}$ by local class field theory. By the Hasse norm principle, the assertion follows. ■

Let p'^n (resp. p'^n) be the minimum annihilator of the group $U_n/(V_n \bar{E}_n)$ (resp. $U_n/(V_n \bar{E}'_n)$). If $U_n/(V_n \bar{E}_n)$ (resp. $U_n/(V_n \bar{E}'_n)$) is not finite, we define $t_n = \infty$ (resp. $t'_n = \infty$).

PROPOSITION 3. For $m \geq n \geq 0$,

$$\#A_m^{\Gamma_n} = \frac{\#A_n \cdot p^{(m-n)(s-1)}}{[E_n : E_n \cap N_{m,n} k_m^\times]} \leq \#A_n \cdot \#(U_n/(V_n \bar{E}_n)),$$

$$\#A_m'^{\Gamma_n} = \frac{\#A'_n \cdot p^{(m-n)(s-1)}}{[E'_n : E'_n \cap N_{m,n} k_m^\times]} \leq \#A'_n \cdot \#(U_n/(V_n \bar{E}'_n)).$$

If $\#(U_n/(V_n \bar{E}_n)) < \infty$ (resp. $\#(U_n/(V_n \bar{E}'_n)) < \infty$) and $m \geq n + t_n$ (resp. $m \geq n + t'_n$), inequality can be replaced with equality.

Proof. By Lemma 4.1 in Chap. 13 of [14], we obtain the first equality. If $U_n/(V_n \bar{E}_n)$ is not finite, the above inequality automatically holds. So assume that $U_n/(V_n \bar{E}_n)$ is finite. Since $S_n = s$, $U_n/V_n \cong \mathbf{Z}_p^{s-1}$. By Lemma 2, $\varepsilon \in E_n \cap N_{m,n} k_m^\times$ if and only if $d_n(\varepsilon^{q-1}) \in \overline{V_n U_n^{p^{m-n}}}$, where q is a large power of p . So $U_n/(V_n \overline{U_n \cap d_n(E_n \cap N_{m,n} k_m^\times)}) = U_n/(V_n U_n^{p^{m-n}}) \cong (\mathbf{Z}/p^{(m-n)}\mathbf{Z})^{s-1}$ for $m \geq n + t_n$. Since

$$\begin{aligned} E_n/(E_n \cap N_{m,n} k_m^\times) &\rightarrow (V_n \bar{E}_n)/(\overline{V_n U_n \cap d_n(E_n \cap N_{m,n} k_m^\times)}) \\ &([\varepsilon] \mapsto [d_n(\varepsilon)^{q-1}]) \end{aligned}$$

is an isomorphism, the first assertion follows. The other assertion can be proved in the same way. ■

THEOREM 1. The following statements are equivalent.

- (1) $A'_\infty = 0$.
- (2) $A'_0 \cong H^1(k_n/k, E'_n)$ and $U_n = V_n \bar{E}'_n$ for some n .

Proof. Using Lemma 1, we obtain the following commutative diagram with exact columns,

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(k_m/k, E'_m) & \rightarrow & A'_0 & \rightarrow & A_m'^{\Gamma} \rightarrow R(k_m/k, E'_m) \rightarrow 0 \\ & & \uparrow \text{Inf}_{n,m}^1 & & \parallel & & \uparrow i_{n,m} \uparrow \text{Inf}_{n,m}^2 \\ 0 & \rightarrow & H^1(k_n/k, E'_n) & \rightarrow & A'_0 & \rightarrow & A_n'^{\Gamma} \rightarrow R(k_n/k, E'_n) \rightarrow 0, \end{array}$$

where $\text{Inf}_{n,m}^1$ maps $[\varepsilon_n]_n$ to $[\varepsilon_n]_m$ and $\text{Inf}_{n,m}^2$ maps $[\varepsilon]_n$ to $[\varepsilon^{p^{m-n}}]_m$.

Assume (1). Then we have $i_{0,n}A'_0 = 0$ and $A'_0 \cong H^1(k_n/k, E'_n)$ for $n \gg 0$ by Proposition 2. Since $\#A'_n$ is bounded as $n \rightarrow \infty$, $\#A_n^{\Gamma}$ is bounded as $n \rightarrow \infty$. This implies that $U_0/(V_0\bar{E}'_0)$ is finite and that

$$\overline{U_0 \cap d_0(E'_0 \cap N_{n,0}k_n^\times)} W_0 V_0 = U_0^{p^n} V_0$$

for $n \geq t'_0$ by Lemma 2. Since $\text{Inf}_{n,m}^2$ is a zero map for $m \gg n$, for all $\varepsilon \in E'_0 \cap N_{n,0}k_n^\times$ there exist some $\varepsilon_m \in E'_m$ with $\varepsilon^{p^{m-n}} = N_{m,0}\varepsilon_m$. Hence we have

$$\overline{U_0 \cap d_0(N_{m,0}E'_m)} W_0 V_0 = U_0^{p^m} V_0$$

for $m \gg n$. Since $N_{m,0} : U_m/V_m \rightarrow (U_0^{p^m} V_0)/V_0$ is an isomorphism, we obtain $U_m = V_m \bar{E}'_m$.

Assume (2). Let n be an integer which satisfies $U_n = V_n \bar{E}'_n$. Since $i_{n,m} : U_n/V_n \rightarrow U_m/V_m$ is an isomorphism (cf. [19]), we have $U_m = V_m \bar{E}'_m$ for $m \geq n$. For all $\varepsilon \in E'_0 \cap N_{n,0}k_n^\times$ there exists $\varepsilon_n \in E'_n$ such that $d_0(\varepsilon(N_{n,0}\varepsilon_n)^{-1}) \in U_0^{p^m} W_0$ for any $m \geq n$. By Lemma 2, $\varepsilon(N_{n,0}\varepsilon_n)^{-1} = \varepsilon' \in E_0 \cap N_{m,0}k_m^\times$. Therefore we have

$$\text{Inf}_{n,m}^2([\varepsilon]_n) = [\varepsilon]_m^{p^{m-n}} = [\varepsilon(N_{n,0}\varepsilon_n)^{-1}]_m^{p^{m-n}} = [\varepsilon']_m^{p^{m-n}}.$$

Since $U_0/(V_0\bar{E}'_0)$ is a quotient of $U_0/(V_0U_0^{p^n})$, $\#A'_n$ and $\#R_n(k_n/k, E'_n)$ are bounded as $n \rightarrow \infty$ by Proposition 3. For $m \gg n$, $\text{Inf}_{n,m}^2$ becomes a zero map and $A_n^{\Gamma} \subseteq H'_n$. Therefore, (2) implies (1) by Proposition 2. ■

As follows from the proof, if $U_n = V_n \bar{E}'_n$ for some n , then it holds for all $n \gg 0$.

COROLLARY 1. *Assume (A) that only one prime ideal in k ramifies in K . The following statements are equivalent.*

- (1) $A'_\infty = 0$.
- (2) $A'_0 \cong H^1(k_n/k, E'_n)$ for some n .

Proof. Since $s = 1$, we have $U_n = V_n$. Therefore, by Theorem 1, the assertion follows. ■

3. TOTALLY REAL CASE

3.1. Theorem

In this section, we assume (B) k is a totally real number field in which p splits completely, and Leopoldt's conjecture is valid for k and p . This

conjecture is valid if and only if $\#A_n^\Gamma$ is bounded as $n \rightarrow \infty$ (cf. Proposition 3). Under the assumption (B), since $D_n \subseteq A_n^\Gamma$, $\#A_n$ is bounded as $n \rightarrow \infty$ if and only if $\#A'_n$ is bounded as $n \rightarrow \infty$. Moreover, Leopoldt's conjecture implies that the cyclotomic \mathbf{Z}_p -extension k_∞ is the unique \mathbf{Z}_p -extension of a totally real number field k , i.e., $K = k_\infty$.

LEMMA 3. Assume (B), then

$$E_0 \cap N_{n+t_0,0} k_{n+t_0}^\times \subseteq \pm E_0^{p^n}$$

for all $n \geq 0$.

Proof. Since p splits completely in k , V_0 is trivial. Leopoldt's conjecture implies that $U_0/(V_0 \bar{E}_0) = U_0/\bar{E}_0$ is finite and that $t_0 < \infty$. For $\varepsilon \in E_0$, $\varepsilon \in N_{n+t_0,0} k_{n+t_0}^\times$ if and only if $d_0(\varepsilon^{p-1}) \in U_0^{p^{n+t_0}}$ (resp. $d_0(\varepsilon^2) \in U_0^{p^{n+t_0+1}}$) for odd prime p (resp. $p = 2$) by Lemma 2. Since $U_0^{p^{t_0}} \subseteq \bar{E}_0$, Leopoldt's conjecture implies that $\varepsilon \in \pm E_0^{p^n}$. ■

In the following lemma, we do not assume (B).

LEMMA 4. Assume that $K = k_\infty$, the cyclotomic \mathbf{Z}_p -extension. For every unit (resp. S -unit) $\varepsilon \in k \cap \mathbf{Q}_\infty$, we have $\varepsilon \in N_{n,0} E_n$ (resp. $\varepsilon \in N_{n,0} E'_n$).

Proof. Put $\mathbf{Q}_{n'} = k \cap \mathbf{Q}_\infty$. Then we have $k_n \cap \mathbf{Q}_\infty = \mathbf{Q}_{n'+n}$. Let us consider an exact sequence of Lemma 1 for \mathbf{Q}_n . By local class field theory and the Hasse norm principle, every unit in \mathbf{Q}_n is a local norm and also a global norm. Since A_n for \mathbf{Q} is trivial for all $n \geq 0$ (cf. [9]), every unit is a norm of some unit. ■

LEMMA 5. Assume (B), then

$$\text{Inf}_{n,m}^2: H^2(k_n/k, E_n) \rightarrow H^2(k_m/k, E_m)$$

is injective for $m \geq n \gg 0$.

Proof. For $\varepsilon \in E_0$, if $\varepsilon^{p^{m-n}} \in N_{m,0} E_m$, we have $\varepsilon \in E_0 \cap N_{n,0} k_n^\times$ by Lemma 2. Therefore it suffices to show that $\text{Inf}_{n,m}^2: R_n \rightarrow R_m$ is injective. Let $m \geq n \geq t_0$. Suppose $\varepsilon \in E_0 \cap N_{m,0} k_m^\times$. Then we have $\varepsilon = \pm \eta^{p^{m-t_0}}$ by Lemma 3. Here we have $\eta^{p^{n-t_0}} \in N_{n,0} k_n^\times$ and $-1 \in N_{m,0} E_m$ by Lemma 4. Hence $\text{Inf}_{n,m}^2: R_n \rightarrow R_m$ is surjective. By Leopoldt's conjecture, $\overline{U_0 \cap d_0(E_0 \cap N_{n,0} k_n^\times)} = U_0^{p^n}$ and $(N_{n,0} E_n)^{p^{m-n}} \subseteq N_{m,0} E_m$ for $m \geq n \geq t_0$. Therefore we have that $\#R_n \geq \#R_m$ and that $\#R_n$ is constant for $n \gg 0$. This implies that $\text{Inf}_{n,m}^2: R_n \rightarrow R_m$ is injective. ■

LEMMA 6. Assume (B), then

$$R(k_n/k, E'_n) \cong \text{Ker}(\text{Inf}_{n, \infty}^2) \oplus R(k_n/k, E_n),$$

$$\text{Ker}(\text{Inf}_{n, \infty}^2) \cong \langle (\varepsilon) \mid \varepsilon \in E'_0 \cap N_{n,0} k_n^\times \rangle / \langle (\varepsilon) \mid \varepsilon \in N_{n,0} E'_n \rangle$$

for $n \gg 0$.

Proof. We first show that there exist two subgroups $\Pi_n, \Pi'_n \subseteq E'_0 \cap N_{n,0} k_n^\times$ such that

$$R(k_n/k, E'_n) \cong (E'_0 \cap N_{n,0} k_n^\times) / N_{n,0} E'_n \cong (\Pi_n / \Pi'_n) \oplus (E_0 \cap N_{n,0} k_n^\times) / N_{n,0} E_n$$

for all $n \gg 0$. By Lemma 5, Leopoldt's conjecture for k and p implies that $\#D_n \leq \#A_n^\Gamma$ is bounded as $n \rightarrow \infty$. Since $N_{m,n} : D_m \rightarrow D_n$ is surjective, $N_{m,n}$ is an isomorphism for all $m \geq n \gg 0$. This implies that $(\Pi') = \langle (\varepsilon) \mid \varepsilon \in N_{n,0} E'_n \rangle \subset I_0$ is constant for all $n \gg 0$. Further, we see that $(\Pi) = \langle (\varepsilon) \mid \varepsilon \in E'_0 \cap N_{n,0} k_n^\times \rangle \subset I_0$ is also constant for all $n \geq t_0$ by Leopoldt's conjecture and Lemma 2. Let

$$(\Pi)/(\Pi') \cong \mathbf{Z}/p^{n_1}\mathbf{Z} \oplus \mathbf{Z}/p^{n_2}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{n_{s'}}\mathbf{Z}$$

as an abelian group. Put $a = \max_{1 \leq i \leq s'} \{n_i\}$ and let a_i be an element in $E'_0 \cap N_{n+a,0} k_{n+a}^\times$ such that $\{(a_i)\}$ is a basis of (Π) and that $\{(a_i^{p^{n_i}})\}$ is a basis of (Π') . Let b_i be an element in $N_{n+a,0} E'_{n+a}$ such that $(b_i) = (a_i^{p^{n_i}})$. For $n \geq t_0$, we have $b_i/a_i^{p^{n_i}} = \pm \varepsilon_i^{p^a}$ for some $\varepsilon_i \in E_0$ by Lemma 3. Put $a'_i = a_i \varepsilon_i^{p^{a-n_i}}$ and $b'_i = a_i^{p^{n_i}} = \pm b_i$. By Lemma 2, we have $\varepsilon_i \in E_0 \cap N_{n,0} k_n^\times$ and $a'_i \in E'_0 \cap N_{n,0} k_n^\times$. By Lemma 4, $-1 \in N_{n,0} E'_n$ and $b'_i \in N_{n,0} E'_n$. Put $\Pi_n = \langle a'_i \rangle_{1 \leq i \leq s'}$ and $\Pi'_n = \langle b'_i \rangle_{1 \leq i \leq s'}$. Then we easily see $E'_0 \cap N_{n,0} E'_n = \Pi_n \oplus (E_0 \cap N_{n,0} E_n)$ and $N_{n,0} E'_n = \Pi'_n \oplus N_{n,0} E_n$. By Lemma 5, $\text{Inf}_{n,m}^2 : R(k_n/k, E_n) \rightarrow R(k_m/k, E_m)$ is an isomorphism for $m \geq n \gg 0$. Therefore we have

$$\begin{aligned} \Pi_n / \Pi'_n &\cong \text{Ker}(\text{Inf}_{n,m}^2 : R(k_n/k, E'_n) \rightarrow R(k_n/k, E'_m)) \\ &= \text{Ker}(\text{Inf}_{n,m}^2 : H^2(k_n/k, E'_n) \rightarrow H^2(k_n/k, E'_m)) \end{aligned}$$

for $m \geq n + a$. ■

THEOREM 2. Assume (B). The following statements are equivalent.

$$(1) \quad A'_\infty = 0.$$

$$(2) \quad A'_0 \cong H^1(k_n/k, E'_n) \quad \text{for some } n \quad \text{and} \quad \text{rk}_p R(k_m/k, E'_m) = \text{rk}_p(R(k_m/k, E'_m)/j_m(R(k_m/k, E_m))) \quad \text{for all } m \gg 0.$$

Proof. Take m and n such that they satisfy the assertion in Lemma 6. Using Lemma 1 and Lemma 6, we obtain the commutative diagram with exact columns,

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^1(k_m/k, E'_m) & \rightarrow & A'_0 & \rightarrow & A_m'^\Gamma & \rightarrow & \text{Ker}(\text{Inf}_{m,\infty}^2) \oplus R_m & \rightarrow & 0 \\ & & \uparrow \text{Inf}_{n,m}^1 & & \parallel & & \uparrow i_{n,m} & & \uparrow \text{Inf}_{n,\infty}^2 & & \\ 0 & \rightarrow & H^1(k_n/k, E'_n) & \rightarrow & A'_0 & \rightarrow & A_n'^\Gamma & \rightarrow & \text{Ker}(\text{Inf}_{n,\infty}^2) \oplus R_n & \rightarrow & 0, \end{array}$$

where $R_n = R(k_n/k, E_n)$.

Assume (1). By Proposition 2, it follows that $A'_0 \cong H^1(k_n/k, E'_n)$ for some n . A map $i_{m,\infty}: A_m'^\Gamma \rightarrow A_\infty'^\Gamma$ is a zero map if and only if $R(k_m/k, E_m)$ is trivial for $m \gg 0$. Therefore $\text{rk}_p R(k_m/k, E'_m) = \text{rk}_p(R(k_m/k, E'_m)/j_m(R(k_m/k, E_m)))$ follows.

Assume (2). By Lemma 6, $R(k_m/k, E_m)$ is trivial for $m \gg 0$. By the above diagram, $i_{m,\infty}: A_m'^\Gamma \rightarrow A_\infty'^\Gamma$ is a zero map for $m \gg 0$. By Proposition 2, the assertion follows. ■

By [5, Theorem 2], $A_\infty = 0$ if and only if A_n^Γ/D_n for $n \gg 0$. By the following proposition and Theorem 2, we can show this assertion.

PROPOSITION 4. *Assume (B) and that $i_{0,n}(A'_0)$ is trivial. For $n \gg 0$,*

$$\begin{aligned} \text{Ker}(H^0(k_n/k, A'_n) \rightarrow H^1(k_n/k, D_n)) &\cong A_n^\Gamma/D_n \cong R(k_n/k, E_n), \\ A_n^\Gamma/(A_n^\Gamma/D_n) &\cong R(k_n/k, E'_n)/j_n(R(k_n/k, E_n)). \end{aligned}$$

Proof. From a short exact sequence $0 \rightarrow D_n \rightarrow A_n \rightarrow A'_n \rightarrow 0$, we have

$$A_n^\Gamma/D_n \cong \text{Ker}(H^0(k_n/k, A'_n) \rightarrow H^1(k_n/k, D_n)).$$

By Lemma 1, we obtain the following exact sequence:

$$0 \rightarrow A_n^\Gamma/(D_n i_{0,n}(A'_0)) \rightarrow R(k_n/k, E_n) \rightarrow 0.$$

Hence by Lemma 6, the assertions immediately follow. ■

3.2. Examples

Let k be a real quadratic field in which p splits. In this case, Leopoldt's conjecture immediately follows for all k and p . If $A'_n = A_n/D_n$ is trivial for all $n \geq 0$, $\#A_n = \#D_n \leq \#A_n^\Gamma$ is bounded as $n \rightarrow \infty$ by Proposition 3.

THEOREM 3. *Let k be a real quadratic field in which p splits. Suppose that A'_n is not trivial for some $n \geq 0$. Then the following statements are equivalent.*

- (1) $A'_\infty = 0$.
- (2) (a) $A'_0 \cong H^1(k_n/k, E'_n)$ for some n ,
- (b) $R(k_m/k, E'_m)$ is cyclic as an abelian group for all $m \gg 0$,
- (c) $R(k_m/k, E'_m)/j_m(R(k_m/k, E'_m))$ is not trivial for all $m \gg 0$.

Proof. Let (II) and (II') be the same groups as in the proof of Lemma 6. By Lemma 4, p is contained in $N_{n,0}E'_n$ for all n . Hence $(II)/(II')$ is a cyclic group. Since Γ and A_n are p -groups, A_n is trivial if and only if A_n^Γ is trivial. For $m \gg 0$, we have $(II)/(II') \cong R(k_m/k, E'_m)/j_m(R(k_m/k, E'_m))$. Therefore the assertion follows by Theorem 2. ■

By Proposition 4, (2) is equivalent to the following statements.

- (a) $A'_0 \cong H^1(k_n/k, E'_n)$ for some n ,
- (b) $A_m'^\Gamma$ is cyclic as an abelian group for all $m \gg 0$,
- (c) $A_m'^\Gamma/(A_m'^\Gamma/D_m)$ is not trivial for all $m \gg 0$.

We will give examples of k to which we can apply Theorem 3.

EXAMPLE 1. Let $k = \mathbf{Q}(\sqrt{2659})$ and $p = 3$. The conjecture was verified for this case in [4]. Following [2], Fukuda and Taya defined invariants $n_0^{(n)}$ and $n_2^{(n)}$ for real quadratic fields k and odd prime numbers p which can be written as

$$p^{n_0^{(n)}} = p^{n+1} \#(U_n/V_n \bar{E}'_n), \quad p^{n_2^{(n)}} = p^{n+1} \#(U_n/V_n \bar{E}_n).$$

From the table of [4], $n_0 = n_0^{(0)} = 2$, $n_2 = n_2^{(0)} = 3$, $n_0^{(1)} = 2$, $n_2^{(1)} = 4$, $\#D_0 = p$, $\#A_0 = p$, $\#D_1 = p$, and $\#A_1 = p^2$. Hence we have $\#A'_0 = 1$, $\#(U_0/V_0 \bar{E}'_0) = p$, and $\#(U_1/V_1 \bar{E}'_1) = 1$. By Theorem 1, we can verify the conjecture for k and p .

By the method in [7], we can verify the conjecture for this field and $p = 3$, using cyclotomic units in k_2 .

EXAMPLE 2. Let $k = \mathbf{Q}(\sqrt{12007})$ and $p = 3$. From the table of [3], $n_0 = 3$, $n_2 = 3$, $n_0^{(1)} = 3$, $n_2^{(1)} = 4$, $\#D_0 = p$, $\#A_0 = p$, $\#D_1 = p$, $\#A_1 = p^2$. Hence we have $\#A'_0 = 1$, $\#(U_0/V_0 \bar{E}'_0) = p^2$, and $\#(U_1/V_1 \bar{E}'_1) = p$. We cannot verify the conjecture for k and p in the same way.

However, by our criterion, we can verify the conjecture for this case in the following way. First we show that A'_n is cyclic as an abelian group for all n . Let ψ be the nontrivial Dirichlet character associated to k and $f_\psi(T)$ the Iwasawa polynomial associated to the p -adic L -function $L_p(s, \psi)$ (see

[10]). Then we see that $f_\psi(T)$ is reducible of degree 2 in $\mathbf{Z}_p[T]$ by computation. By the Iwasawa main conjecture proved in [16], $\text{Gal}(M/k_\infty)$ is isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$ as an abelian group, where M is the maximal abelian p -extension of k unramified outside p (cf. [19]). Moreover, since $A_0 = D_0 \cong \mathbf{Z}/p\mathbf{Z}$ and A'_n is a quotient of $\text{Gal}(M/k_\infty)$, A'_n is cyclic as an abelian group. Further, since $A_n'^{F_1} \cong A_n^{F_1}/D_n$, $\#D_n \geq \#A_n^{F_1}/\#A_n'^{F_1} = p^4/p^2 = p^2$ for $n \gg 1$ by Proposition 3. Hence we have $\#A_n'^F = p^2 \geq \#(A_n^F/D_n) \geq p^3/p^2 = p$ for $n \gg 1$. Therefore (a), (b), and (c) hold for k and p .

By the method in [7], we can verify the conjecture for this field and $p=3$, using cyclotomic units in k_3 .

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